A 2-D Composite Polygonal Mixed Finite Element

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Abstract

General hexahedral and quadrangular grids present a challenge for mixed finite elements for second-order, elliptic problems. We define and analyze a mixed finite element method for a mesh made up of star-shaped polygons. The scalar unknown is approximated by element-wise constants and the vector unknown is determined by its flux through the edges of the polygons. The elements are composite elements. Each polygon is split into triangles by taking an interior point of the polygon, one for which it is star-shaped, and considering the triangles radiating from that point and having one side as a side of the polygon. Convergence of the method is proven, and numerical experiments are shown to confirm the theoretical results.

Keywords: mixed finite element, polygonal mesh, flow in porous media

1 Introduction

Single-phase, incompressible flow in a porous medium is governed by the Darcy flow equation, a second-order elliptic equation, which when written in mixed form as a system of first order equations consists of a conservation equation together with Darcy's law. If gravity is neglected, these equations may be written as follows:

$$\nabla\cdot \mathbf{u} = f \quad \text{and} \quad \mathbf{u} = -\mathbf{K}\,\nabla\, p,$$

where p is the fluid pressure, $\mathbf{u}$ is the Darcy flow velocity, the coefficient $\mathbf{K}$ is a symmetric, positive-definite tensor, and f is a source term. It has been known since the early 1980's that mixed finite element methods are particularly well suited to solving these equations numerically. In particular, a mixed method is locally conservative, it calculates the Darcy velocity $\mathbf{u}$ simultaneously with the pressure p and to the same order of accuracy, it is well adapted to handling a highly discontinuous and non diagonal permeability tensor $\mathbf{K}$. For most

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applications it is desirable to have the discretization of the domain into finite elements conform to the layering of the domain by the permeability coefficient $\mathbf{K}$. This is of course easily done with a grid of triangles or tetrahedra. However there are obvious advantages to using a logically rectangular grid, and for a grid of rectangles or rectangular solids, adapting to the natural layering of the domain leads to a deformation of the rectangular structure.

Several mixed finite element methods for second-order elliptic problems, both for meshes of triangles or tetrahedra and for meshes of rectangles or rectangular solids have been introduced and analyzed. The most well known of these are probably the elements defined by Raviart and Thomas (and by Nédélec in 3D) [15, 13] and the elements defined by Brezzi, Douglas and Marini (and with Fortin for rectangular solids and by Brezzi, Douglas, Duran and Fortin for tetrahedra), [**?**, **?**, **?**]. Methods have also been developed for meshes of parallelograms or parallelipipeds and for triangular prisms. See also [6] or [16] for an extensive bibliography. However straight forward extensions of these methods to handle meshes of quadrilateral polygons or hexahedra lack essential approximation properties. Several articles have addressed the problem of defining a mixed finite element on a mesh of quadrilaterals or hexahedra. Some of these, such as [18, 3, 9, 19, 1], have constructed mixed methods by enriching the polynomial approximation space, but the number of degrees of freedom can quickly become unmanageable, particularly in 3 dimensions. Others, such as [11, 12, 17], have instead kept the original degrees of freedom of the lowest order Raviart-Thomas-(Nédélec) elements, but, following an idea introduced by Kuznetsov and Repin in [11], have used composite elements. In [17], each hexahedron is divided into 5 tetrahedra, each face being divided into 2 triangles. The method has optimal convergence properties, but there is no evident way to extend the method to the case of generalized hexahedra which might have a non planar face. To overcome this problem, a composite element, in which a deformed cube is divided into 24 tetrahedra, was introduced in [5]. This element, obtained by dividing each of the 6 faces of the deformed cube into 4 triangles and considering the cones over the 24 resulting triangles emanating from an interior point of the deformed cube, has in addition the following desirable attributes: it has good symmetry properties and the division into 24 tetrahedra is uniquely defined.

The aim of the present article is to analyze the two-dimensional counterpart of the composite element of [5]. The results of [11, 12, 17] are extended for the 2D setting to the case of a composite element whose division into subcells has a vertex in the interior of the cell. This we view as a first step towards the analysis of the 3D composite element of [5].

The remainder of this article is organized as follows: Section 2 recalls some of the basic theory for mixed finite element methods. In Section 3, the composite method is defined for a mesh made up of polygons. Approximation spaces are defined locally on each polygonal cell using a triangular submesh constructed by adding an additional vertex inside the cell. With this additional point in the interior of a cell E, the local approximation space contains nontrivial velocity vector fields in the image of the mapping $\mathrm{curl} : H^1(E) \to \mathbf{H}(\mathrm{div}, E)$ and thus in the kernel of div : $\mathbf{H}(\mathrm{div}, E) \to L^2(E)$. Section 4 highlights the presence of these velocities, which must be taken into account. Section 5 gives some preliminary results for this extra difficulty. Optimal order convergence of the method is proven in Section 6, and numerical experiments corroborating this result are shown in Section 7.

2 Numerical analysis for mixed methods

Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain that represents the porous medium. The equations that govern an incompressible Darcy flow may be written as follows:

$$\begin{aligned} \mathbf{u} &= -\mathbf{K}\,\nabla p && \text{in } \Omega, \\ \nabla\cdot \mathbf{u} &= f && \text{in } \Omega, \\ p &= p_{\mathrm{d}} && \text{on } \partial\Omega, \end{aligned}$$

where the unknowns are the pressure p and the Darcy velocity $\mathbf{u}$. The symmetric, positive-definite tensor $\mathbf{K}$ models the diffusion of the fluid in the porous medium. The function $f : \Omega \to \mathbb{R}$ is a source term. For simplicity, we have assumed that the boundary conditions are only Dirichlet conditions. Let $\mathcal{M} = L^2(\Omega)$ and $\mathcal{W} = \mathbf{H}(\mathrm{div}, \Omega)$. The weak mixed formulation of the system is then

$$\begin{aligned} &\text{Find } \mathbf{u} \in \mathcal{W} \text{ and } p \in \mathcal{M} \text{ such that} \\ &a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = \mathcal{L}_{\mathcal{W}}(\mathbf{v}), \quad \forall \mathbf{v} \in \mathcal{W}, \\ &b(\mathbf{u}, q) = \mathcal{L}_{\mathcal{M}}(q), \qquad\qquad\quad \forall q \in \mathcal{M}, \end{aligned} \tag{$\mathcal{P}$}$$

where the bilinear forms $a : \mathcal{W} \times \mathcal{W} \to \mathbb{R}$ and $b : \mathcal{W} \times \mathcal{M} \to \mathbb{R}$ are defined by

$$a(\mathbf{u}, \mathbf{v}) = \int_\Omega \mathbf{K}^{-1}\mathbf{u} \cdot \mathbf{v}, \qquad \forall \mathbf{u} \in \mathcal{W}, \forall \mathbf{v} \in \mathcal{W}, \tag{1}$$

$$b(\mathbf{u}, q) = -\int_\Omega q\,\nabla\cdot \mathbf{u}, \qquad \forall \mathbf{u} \in \mathcal{W}, \forall q \in \mathcal{M}, \tag{2}$$

and the linear forms $\mathcal{L}_{\mathcal{W}} : \mathcal{W} \to \mathbb{R}$, and $\mathcal{L}_{\mathcal{M}} : \mathcal{W} \to \mathbb{R}$ by

$$\mathcal{L}_{\mathcal{W}}(\mathbf{v}) = -\int_{\partial\Omega} p_{\mathrm{d}}\mathbf{v} \cdot \mathbf{n}, \qquad \forall \mathbf{v} \in \mathcal{W},$$

$$\mathcal{L}_{\mathcal{M}}(q) = -\int_\Omega f q, \qquad \forall q \in \mathcal{M},$$

where p_{d} is a function on the boundary of Ω determined by the Dirichlet data. In [8, Proposition 3.1], [6] and [16], it is shown that if the bilinear forms a and b defined respectively in (1) and (2) satisfy the following conditions:

1. a is $\mathcal{V}$-elliptic, where $\mathcal{V} = \{\mathbf{v} \in \mathcal{W} : b(\mathbf{v}, q) = 0, \forall q \in \mathcal{M}\}$; i.e.

$$\exists \alpha > 0 \text{ such that} \quad a(\mathbf{v}, \mathbf{v}) \geqslant \alpha \|\mathbf{v}\|^2_{\mathbf{H}(\mathrm{div},\Omega)}, \quad \forall \mathbf{v} \in \mathcal{V}, \tag{3a}$$

2. b satisfies the following inf-sup condition:

$$\exists \beta > 0 \text{ such that } \quad \inf_{q \in \mathcal{M}} \sup_{\mathbf{v} \in \mathcal{W}} b(\mathbf{v}, q) \geqslant \beta \|\mathbf{v}\|_{\mathbf{H}(\mathrm{div},\Omega)} \|q\|_{0,\Omega}, \tag{3b}$$

then problem $(\mathcal{P})$ admits a unique solution $(\mathbf{u}, p)$.

We consider now a discrete version of problem $(\mathcal{P})$. Let $\mathcal{M}_h \subset \mathcal{M}$ denote an approximation space for the pressure and $\mathcal{W}_h \subset \mathcal{W}$ an approximation space for the velocity. The discrete problem $(\mathcal{P})$ obtained by replacing the spaces $\mathcal{W}$ and $\mathcal{M}$ by the finite dimensional spaces $\mathcal{W}_h$ and $\mathcal{M}_h$, respectively in $(\mathcal{P})$ is

$$\begin{aligned}
&\text{Find } \mathbf{u}_h \in \mathcal{W}_h \text{ and } p_h \in \mathcal{M}_h \text{ such that} \\
&a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) = \mathcal{L}_{\mathcal{W}}(\mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathcal{W}_h, \\
&b(\mathbf{u}_h, q_h) = \mathcal{L}_{\mathcal{M}}(q_h), \qquad\qquad\qquad \forall q_h \in \mathcal{M}_h.
\end{aligned} \tag{$\mathcal{P}_h$}$$

As for the continuous problem $(\mathcal{P})$, if for the approximation spaces $\mathcal{M}_h$ and $\mathcal{W}_h$ the bilinear forms a and b defined in (1) and (2) satisfy the following conditions:

1. a is $\mathcal{V}_h$-elliptic, where $\mathcal{V}_h = \{\mathbf{v}_h \in \mathcal{W}_h : b(\mathbf{v}_h, q_h) = 0, \forall q_h \in \mathcal{M}_h\}$; i.e.

$$\exists \alpha_h > 0 \text{ such that } \quad a(\mathbf{v}_h, \mathbf{v}_h) \geqslant \alpha_h \|\mathbf{v}_h\|^2_{\mathbf{H}(\mathrm{div},\Omega)}, \quad \forall \mathbf{v}_h \in \mathcal{V}_h, \tag{4a}$$

2. b satisfies the following discrete inf-sup condition:

$$\exists \beta_h > 0 \text{ such that, } \quad \inf_{q_h \in \mathcal{M}_h} \sup_{\mathbf{v}_h \in \mathcal{W}_h} b(\mathbf{v}_h, q_h) \geqslant \beta_h \|\mathbf{v}_h\|_{\mathbf{H}(\mathrm{div},\Omega)} \|q_h\|_{0,\Omega}, \tag{4b}$$

then the problem $(\mathcal{P}_h)$ admits a unique solution $(\mathbf{u}_h, p_h)$, and further there exists $C > 0$ depending only on the constants of continuity of a and b and the constants α_h and β_h such that

$$\begin{aligned}
&\|p - p_h\|_{0,\Omega} + \|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{H}(\mathrm{div},\Omega)} \\
&\qquad \leqslant C \left(\inf_{q_h \in \mathcal{M}_h} \|p - q_h\|_{0,\Omega} + \inf_{\mathbf{v}_h \in \mathcal{W}_h} \|\mathbf{u} - \mathbf{v}_h\|_{\mathbf{H}(\mathrm{div},\Omega)} \right).
\end{aligned} \tag{5}$$

Thus if the constants α_h and β_h can be chosen independently of h, then C will also be independent of h, and the problem of obtaining error estimates is then reduced to a problem of interpolation.

In the following we construct a composite mixed finite element space satisfying conditions (4a) and (4b) with constants independent of the discretization parameter h. Interpolation and approximation errors are shown later in Section 6.

3 A composite method for polygons

In this section we define the finite dimensional spaces $\mathcal{W}_h \subset \mathcal{W}$ and $\mathcal{M}_h \subset \mathcal{M}$ in which the approximations $\mathbf{u}_h$ of $\mathbf{u}$ and p_h of p will be sought. These are two dimensional analogues of the spaces defined in [5].

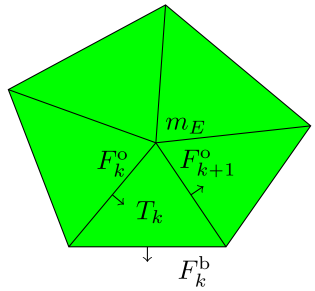

Figure 1: The mesh $\mathcal{T}_E$ for a pentagon E divided into 5 triangles

Let $\mathcal{T}_h$ be a conforming discretization of the domain Ω, made up of polygons E of diameter no greater than h. Each polygonal cell $E \in \mathcal{T}_h$ is assumed to be star-shaped with respect to the barycenter m_E of its set of vertices. To define the composite elements we make use of a refinement $\widetilde{\mathcal{T}}_h$ of $\mathcal{T}_h$ made up of triangles. If $E \in \mathcal{T}_h$ is a polygon with n_E edges, then it is divided into n_E triangles each of which is the cone with summit m_E over one of the edges of E as shown in Figure 1. The set of these n_E triangles is denoted $\widetilde{\mathcal{T}}_E$, and $\widetilde{\mathcal{T}}_h$ is defined by

$$\widetilde{\mathcal{T}}_h = \bigcup_{E \in \mathcal{T}_h} \widetilde{\mathcal{T}}_E.$$

We let $\mathcal{F}_h$ denote the set of all edges of elements of $\mathcal{T}_h$ and $\widetilde{\mathcal{F}}_h$ the set of all edges of elements of $\widetilde{\mathcal{T}}_h$. Then $\widetilde{\mathcal{F}}_E$ will denote the set of edges of elements of $\widetilde{\mathcal{T}}_E$.

We will also use some intermediate approximation spaces associated with the mesh $\widetilde{\mathcal{T}}_h$

$$\widetilde{\mathcal{M}}_h = \{\widetilde{q} \in \mathcal{M} \colon \widetilde{q}|_T \text{ is constant on } T, \ \forall T \in \widetilde{\mathcal{T}}_h\}, \qquad \widetilde{\mathcal{W}}_h = \mathbf{RT}_0(\widetilde{\mathcal{T}}_h),$$

and the following spaces associated with the triangular submesh $\widetilde{\mathcal{T}}_E$:

$$\widetilde{\mathcal{M}}_E = \{\widetilde{q} \in L^2(E) \colon \widetilde{q}|_T \text{ is constant on } T, \ \forall T \in \widetilde{\mathcal{T}}_E\}, \quad \widetilde{\mathcal{W}}_E = \mathbf{RT}_0(\widetilde{\mathcal{T}}_E),$$

where, for any triangular finite element mesh $\widetilde{\mathcal{T}}$, $\mathbf{RT}_0(\widetilde{\mathcal{T}})$ denotes the Raviart-Thomas space of lowest order associated with $\widetilde{\mathcal{T}}$.

The approximation space for the pressure, $\mathcal{M}_h \subset \widetilde{\mathcal{M}}_h \subset \mathcal{M}$, is defined, just as in the case of the standard Raviart-Thomas method with lowest-order elements, to be the space of functions which are constant on each polygon $E \in \mathcal{T}_h$:

$$\mathcal{M}_h = \{q \in \widetilde{\mathcal{M}}_h \subset \mathcal{M} \colon q|_E \text{ is constant on } E, \ \forall E \in \mathcal{T}_h\}.$$

We would like to define the approximation space for the velocity, $\mathcal{W}_h \subset \widetilde{\mathcal{W}}_h \subset \mathcal{W}$ such that, just as in the case of the standard Raviart-Thomas method, for each $\mathbf{v}_h \in \mathcal{W}_h$,

i) $\nabla \cdot \mathbf{v}_h$ is constant on each cell $E \in \mathcal{T}_h$; i.e. $\nabla \cdot \mathbf{v}_h \in \mathcal{M}_h$.

ii) $\mathbf{v}_h$ is defined uniquely by its (constant) normal components on the edges of the mesh $\mathcal{T}_h$.

The space $\mathcal{W}_h$ will be defined locally; i.e. for each $E \in \mathcal{T}_h$, we will define a space $\mathcal{W}_E \subset \widetilde{\mathcal{W}}_E$, and then $\mathcal{W}_h$ will be defined by

$$\mathcal{W}_h = \{\mathbf{v} \in \widetilde{\mathcal{W}}_h \subset \mathcal{W} : \mathbf{v}|_E \in \mathcal{W}_E,\ \forall E \in \mathcal{T}_h\}.$$

To define the spaces $\mathcal{W}_E$ we introduce some more notations. Let E be a polygon with n_E edges in $\mathcal{T}_h$. Then as noted earlier $\widetilde{\mathcal{T}}_E$ has n_E triangles, $T_1, \ldots, T_{n_E}$. The set $\widetilde{\mathcal{F}}_E$ of edges of these triangles contains $2n_E$ elements, n_E edges $F_1^{\text{ext}}, \ldots, F_{n_E}^{\text{ext}}$ on the boundary of E, and n_E edges $F_1^{\text{int}}, \ldots, F_{n_E}^{\text{int}}$ in the interior of E. We denote the corresponding sets of edges as follows:

$$\mathcal{F}_E^{\text{ext}} = \{F \in \widetilde{\mathcal{F}}_E : F \subset \partial E\} = \widetilde{\mathcal{F}}_E \bigcap \mathcal{F}_h,$$
$$\mathcal{F}_E^{\text{int}} = \{F \in \widetilde{\mathcal{F}}_E : F \subset E^{\text{o}}\} = \widetilde{\mathcal{F}}_E \backslash \mathcal{F}_E^{\text{ext}}.$$

We suppose that these triangles and edges are numbered such that (see Figure 1):

- $F_1^{\text{int}} \in \mathcal{F}_E^{\text{int}}$ is an edge of T_1 and of T_{n_E}.
- $F_k^{\text{int}} \in \mathcal{F}_E^{\text{int}}$ is an edge of T_{k-1} and of T_k, $k = 2, \ldots, n_E$.
- $F_k^{\text{ext}} \in \mathcal{F}_E^{\text{ext}}$ is an edge of T_k, $k = 1, \ldots, n_E$.

For each edge F in $\widetilde{\mathcal{F}}_E$ we choose a unit normal vector $\mathbf{n}_F$ such that if $F \in \mathcal{F}_E^{\text{ext}}$ then $\mathbf{n}_F$ points outward from E, and if $F = F_k^{\text{int}} \in \mathcal{F}_E^{\text{int}}$ then $\mathbf{n}_{F_k^{\text{int}}}$ points inward toward T_k; again see Figure 1.

Clearly, if **??** is to be satisfied, the dimension of $\mathcal{W}_E$ should be equal to n_E, the number of edges of E, whereas that of $\widetilde{\mathcal{W}}_E$ is $2n_E$, the number of edges in $\widetilde{\mathcal{F}}_E$. If $\mathbf{v} \in \widetilde{\mathcal{W}}_E$, then, according to the divergence theorem, the flux through the edges of T_k are related by the equations

$$v_{k+1}^{\text{int}} - v_k^{\text{int}} = d_k - v_k^{\text{ext}}, \quad k \in \mathbb{Z}_{n_E},$$

where v_k^{int} and v_k^{ext}, are the fluxes of $\mathbf{v}$ through F_k^{int} and F_k^{ext} respectively, and d_k is the integral of the divergence of $\mathbf{v}$ over T_k, $k = 1, \ldots, n_E$. If the n_E values v_k^{ext} are known then the average value (over E) of the divergence is known (again from the divergence theorem), and if **??** is satisfied, then so are the n_E values d_k. From (**??**), we have then n_E equations in n_E unknowns which are the fluxes on internal edges, however they are clearly not independent: summing these n_E equations we eliminate the unknowns and obtain the divergence theorem for E which we have already used. Clearly if $\{v_k^{\text{int}} : k = 1, \ldots, n_E\}$ is a solution then so is $\{v_k^{\text{int}} + c : k = 1, \ldots, n_E\}$ for any constant c; i.e. there remains one extra dimension, in particular that generated by the nontrivial element $\boldsymbol{\varphi} \in \widetilde{\mathcal{W}}_E$ with $\operatorname{div} \boldsymbol{\varphi} = 0$ defined by

$$\varphi_k^{\text{ext}} = \int_{F_k^{\text{ext}}} \boldsymbol{\varphi} \cdot \mathbf{n}_{F_k^{\text{ext}}} = 0, \qquad \varphi_k^{\text{int}} = \int_{F_k^{\text{int}}} \boldsymbol{\varphi} \cdot \mathbf{n}_{F_k^{\text{int}}} = 1, \qquad k = 1, \ldots, n_E.$$

However any $n_E - 1$ of these equations are independent and to obtain a solution it would suffice to fix v_k^{int} for any k, and to avoid a rotation it seems reasonable to set some $v_k^{\mathrm{int}} = 0$. However this leaves an arbitrary choice so instead we require that the average value of the v_k^{int}'s be 0. So we define the space $\mathcal{W}_E$ by

$$\mathcal{W}_E = \{\mathbf{v} \in \widetilde{\mathcal{W}}_E : \operatorname{div}\mathbf{v} \text{ is constant on } E \text{ and } \phi_E(\mathbf{v}) = 0\},$$

where, for $\mathbf{v} \in \widetilde{\mathcal{W}}_E$, $\phi_E(\mathbf{v})$ is defined by

$$\phi_E(\mathbf{v}) = \frac{1}{n_E} \sum_{k=1}^{n_E} \int_{F_k^{\mathrm{int}}} \mathbf{v} \cdot \mathbf{n}_{F_k^{\mathrm{int}}}, \quad \forall \mathbf{v} \in \widetilde{\mathcal{W}}_E. \tag{6}$$

Then the local problems $(\mathcal{P}_{E,F})$ which are used to compute the basis functions for $\mathcal{W}_E$ are, for all $F \in \mathcal{F}_E^{\mathrm{ext}}$,

$$\begin{aligned}
&\text{Find } \mathbf{w}_{E,F} \in \widetilde{\mathcal{W}}_E \text{ such that} \\
&\quad \nabla \cdot \mathbf{w}_{E,F} = \frac{1}{|E|}, \\
&\mathbf{w}_{E,F} \cdot \mathbf{n}_{F'} = \frac{1}{|F|}\delta_F^{F'} \quad \forall F' \in \mathcal{F}_E^{\mathrm{ext}}, \\
&\phi_E(\mathbf{w}_{E,F}) = 0.
\end{aligned} \tag{$\mathcal{P}_{E,F}$}$$

An explicit formula to compute the normal components of velocities from the conditions defined in $(\mathcal{P}_{E,F})$ is given in Lemma 4.1. Consequently, the problem $(\mathcal{P}_{E,F})$ is well-posed, and has a unique solution.

Remark 1. Of course we could define basis functions $\mathbf{w}'_{E,F}$ and the space that they generate $\mathcal{W}'_E$ by imposing, instead of the requirement that $\phi_E(\mathbf{w}'_{E,F}) = 0$, the requirement that $\mathbf{w}'_{E,F}$ be a discrete gradient in the sense that it is a solution of the following problem

$$\begin{aligned}
&\text{Find } \mathbf{w}'_{E,F} \in \widetilde{\mathcal{W}}_E \text{ and } p_{E,F} \in \widetilde{\mathcal{M}}_E \text{ such that} \\
&\hat{a}(\mathbf{w}'_{E,F}, \tilde{\mathbf{v}}) + b(\tilde{\mathbf{v}}, p_{E,F}) = 0 \qquad \forall \tilde{\mathbf{v}} \in \widetilde{\mathcal{W}}_E, \\
&b(\mathbf{w}'_{E,F}, \tilde{q}) = -\frac{1}{|E|}\int_E \tilde{q} \qquad \forall \tilde{q} \in \widetilde{\mathcal{M}}_E, \\
&\mathbf{w}'_{E,F} \cdot \mathbf{n}_{F'} = \frac{1}{|F|}\delta_F^{F'} \qquad \forall F' \in \mathcal{F}_E^{\mathrm{ext}}, \\
&\int_E p_{E,F} = 0,
\end{aligned}$$

where the bilinear form $\hat{a} : \mathcal{W} \times \mathcal{W} \to \mathbb{R}$ is defined by $\hat{a}(\mathbf{u}, \mathbf{v}) = \int_E \mathbf{u} \cdot \mathbf{v}$, for $\mathbf{u} \in \widetilde{\mathcal{W}}_E$ and $\mathbf{v} \in \widetilde{\mathcal{W}}_E$.

These definitions coincide in the case that the polygon E is a regular quadrangle, coming from a linear transformation of the reference element. We compared the methods resulting from these two choices of basis functions and could not observe any significant difference in the error behavior. However the computing cost is lower when using $(\mathcal{P}_{E,F})$ because unlike $(\mathcal{P}'_{E,F})$, the solution of this problem does not depends on the shape of the polygon.

4 An explicit expression for the flux across the interior edges of a composite element

We know that a velocity $\mathbf{v} \in \widetilde{\mathcal{W}}_E$ is uniquely defined by its normal components. An expression is given for the interior fluxes, which depends only on the divergence of $\mathbf{v}$, its normal components at the boundary, and $\phi_E(\mathbf{v})$ defined in (6). This expression of the normal components is used later to decompose $\mathbf{v}$ with a velocity rotating around the node m_E, and a remainder.

Lemma 4.1. *Let $\mathbf{v} \in \widetilde{\mathcal{W}}_E$ be arbitrary. Then for all $k = 1, \dots, n_E$, we have*

$$v_k^{\text{int}} = \sum_{i=1}^{k-1} \frac{i}{n_E} \left(d_i - v_i^{\text{ext}}\right) - \sum_{i=k}^{n_E-1} \frac{n_E - i}{n_E} \left(d_i - v_i^{\text{ext}}\right) + \phi_E(\mathbf{v}), \tag{7}$$

where $\phi_E(\mathbf{v})$ is defined in (6)*, and*

$$d_i = \int_{T_i} \nabla\cdot \mathbf{v}, \quad v_i^{\text{int}} = \int_{F_i^{\text{int}}} \mathbf{v} \cdot \mathbf{n}_{F_i^{\text{int}}}, \quad v_i^{\text{ext}} = \int_{F_i^{\text{ext}}} \mathbf{v} \cdot \mathbf{n}_{F_i^{\text{ext}}}, \quad \textit{for } i = 1, \dots, n_E.$$

Proof. Let $\mathbf{v} \in \widetilde{\mathcal{W}}_E$ and $k \in \{1, \dots, n_E\}$. To prove (7), we introduce the divergence theorem in (**??**) for $\mathbf{v}$ on triangles T_i for $i = 1, \dots, n_E - 1$. Counting the definition of $\phi_E(\mathbf{v})$ in (6), there is n_E equations, which are written in matrix form and where the definition of $\phi_E(\mathbf{v})$ is placed on the k^{th} line. We obtain the linear system

$$\mathbf{M_k} \mathbf{v}^{\text{int}} = \mathbf{b_k},$$

where the components of the unknown vector $\mathbf{v}^{\text{int}}$ are its normal components to the internal edges, $v_i^{\text{int}} = \int_{F_i^{\text{int}}} \mathbf{v} \cdot \mathbf{n}_{F_i^{\text{int}}}$. The matrix is

$$\mathbf{M_k} = \begin{array}{c} \begin{pmatrix} -1 & 1 & & & & & \\ & \ddots & \ddots & & & & \\ & & -1 & 1 & & & \\ 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ & & & -1 & 1 & & \\ & & & & \ddots & \ddots & \\ & & & & & -1 & 1 \end{pmatrix} \end{array} \begin{array}{l} \\ \\ \\ \leftarrow \text{ line } k, \\ \\ \\ \\ \end{array}$$

and the righthand side is

$$\begin{aligned} b_{k\,i} &= d_i - v_i^{\text{ext}} && \text{for } i < k, \\ b_{k\,k} &= n_E \phi_E(\mathbf{v}) && \text{for } i = k, \\ b_{k\,i} &= d_{i-1} - v_{i-1}^{\text{ext}} && \text{for } i > k. \end{aligned}$$

Moreover, we can find an expression of the normal component v_k^{int} using the structure of the matrix, by summing the lines from 1 to $k-1$ to the k^{th} line, and subtracting the lines from $k+1$ to n_E. On one hand we have

$$\sum_{i=1}^{k-1} i b_{ki} = \sum_{i=1}^{k-1} i\left(d_i - v_i^{\text{ext}}\right) = \sum_{i=1}^{k-1} i\left(v_{i+1}^{\text{int}} - v_i^{\text{int}}\right) = k v_k^{\text{int}} - \sum_{i=1}^{k} v_i^{\text{int}}.$$

and on the other hand we have

$$\begin{aligned}\sum_{i=k+1}^{n_E} (i-1-n_E) b_{ki} &= \sum_{i=k}^{n_E-1} (i-n_E)\left(d_i - v_i^{\text{ext}}\right)\\ &= \sum_{i=k}^{n_E-1} (i-n_E)\left(v_{i+1}^{\text{int}} - v_i^{\text{int}}\right)\\ &= (n_E-k) v_k^{\text{int}} - \sum_{i=k+1}^{n_E} v_i^{\text{int}}.\end{aligned}$$

Therefore,

$$b_{kk} + \sum_{i=1}^{k-1} i b_{ki} + \sum_{i=k+1}^{n_E} (i-1-n_E) b_{ki} = n_E \phi_E(\mathbf{v}) + n_E v_k^{\text{int}} - \sum_{i=1}^{n_E} v_i^{\text{int}}.$$

And using the definition of $\phi_E(\mathbf{v})$ in (6), it follows that

$$v_k^{\text{int}} = \frac{1}{n_E}\left(b_{kk} + \sum_{i=1}^{k-1} i b_{ki} + \sum_{i=k+1}^{n_E} (i-1-n_E) b_{ki}\right),$$

which can be rewritten as (7). □

Definition 1. According to (7), any velocity $\mathbf{v} \in \widetilde{\mathcal{W}}_E$ can be split into a velocity rotating around the interior point of E denoted by $\mathbf{\Phi}_E(\mathbf{v}) \in \widetilde{\mathcal{W}}_E$ and a remainder $\mathbf{\Psi}_E(\mathbf{v}) \in \widetilde{\mathcal{W}}_E$:

$$\mathbf{v} = \mathbf{\Phi}_E(\mathbf{v}) + \mathbf{\Psi}_E(\mathbf{v}). \tag{8}$$

Both projections are defined by their normal components on interior edges $F_k^{\text{int}} \in \mathcal{F}_E^{\text{int}}$ and on edges included in the boundary of the mesh $F_k^{\text{ext}} \in \mathcal{F}_E^{\text{ext}}$. $\mathbf{\Phi}_E(\mathbf{v})$ is defined by using the definition of $\phi_E(\mathbf{v})$ in (6):

$$\int_{F_k^{\text{int}}} \mathbf{\Phi}_E(\mathbf{v}) \cdot \mathbf{n}_{F_k^{\text{int}}} = \phi_E(\mathbf{v}), \quad \int_{F_k^{\text{ext}}} \mathbf{\Phi}_E(\mathbf{v}) \cdot \mathbf{n}_{F_k^{\text{ext}}} = 0, \quad k = 1, \ldots, n_E,$$

and the remainder $\mathbf{\Psi}_E(\mathbf{v})$:

$$\begin{aligned}\int_{F_k^{\text{int}}} \mathbf{\Psi}_E(\mathbf{v}) \cdot \mathbf{n}_{F_k^{\text{int}}} &= \sum_{i=1}^{k-1} \frac{i}{n_E} \left(\int_{T_i} \nabla\cdot \mathbf{v} - \int_{F_i^{\text{ext}}} \mathbf{v} \cdot \mathbf{n}_{F_i^{\text{ext}}} \right) \\ &\quad - \sum_{i=k}^{n_E-1} \frac{n_E - i}{n_E} \left(\int_{T_i} \nabla\cdot \mathbf{v} - \int_{F_i^{\text{ext}}} \mathbf{v} \cdot \mathbf{n}_{F_i^{\text{ext}}} \right), \\ \int_{F_k^{\text{ext}}} \mathbf{\Psi}_E(\mathbf{v}) \cdot \mathbf{n}_{F_k^{\text{ext}}} &= \int_{F_k^{\text{ext}}} \mathbf{v} \cdot \mathbf{n}_{F_k^{\text{ext}}}, \qquad k = 1, \dots, n_E.\end{aligned} \tag{9}$$

5 A bound on the velocities

From the Definition 1, we can study the behavior of a velocity $\mathbf{v} \in \widetilde{\mathcal{W}}_E$ inside the composite element. This decomposition allow us to estimate the norm of $\mathbf{v}$, from its divergence, its normal components at the boundary and $\phi_E(\mathbf{v})$.

5.1 A bound of the remainder

We need first to introduce the following norm for functions of $\widetilde{\mathcal{W}}_h$

$$[\![\mathbf{v}]\!]_T = \int_{\partial T} |\mathbf{v} \cdot \mathbf{n}|, \quad \forall T \in \widetilde{\mathcal{T}}_h, \quad \forall \mathbf{v} \in \widetilde{\mathcal{W}}_h,$$

and a shape regularity assumption to estimate the $\mathbf{L}^2$ norm of $\mathbf{v}$.

Definition 2 (Shape regularity)**.** Let ρ_T be the radius of the inscribed circle of the triangle T and h_T be its diameter. The shape constant of T is $\sigma_T = \frac{h_T}{\rho_T}$. σ_E, the shape constant of the mesh $\widetilde{\mathcal{T}}_E$ and σ_h, that of the mesh $\mathcal{T}_h$ are

$$\sigma_E = \max_{T \in \widetilde{\mathcal{T}}_E} \sigma_T, \qquad \sigma_h = \max_{E \in \mathcal{T}_h} \sigma_E = \max_{T \in \widetilde{\mathcal{T}}_h} \sigma_T.$$

The family of meshes $\{\mathcal{T}_h, h \in \mathcal{H}\}$ is shape regular if σ_h is uniformly bounded.

Lemma 5.1 (Norm equivalence)**.** *For any velocity $\mathbf{v} \in \mathbf{RTN}_0(T)$, there are constants α_T and β_T such that*

$$\alpha_T \leqslant \frac{\|\mathbf{v}\|_{0,T}^2}{[\![\mathbf{v}]\!]_T^2} \leqslant \beta_T,$$

with α_T and β_T non-negative constants that depend on T.

Moreover, if the family of meshes $\{\mathcal{T}_h, h \in \mathcal{H}\}$ is shape regular, then there are constants α and β independent of T and h such that $\forall T \in \widetilde{\mathcal{T}}_E$, $E \in \mathcal{T}_h$, $h \in \mathcal{H}$

$$\alpha \leqslant \frac{\|\mathbf{v}\|_{0,T}^2}{[\![\mathbf{v}]\!]_T^2} \leqslant \beta. \tag{10}$$

Proof. The proof is a scaling argument. It is similar to that in [17] where E was an hexahedron divided into 5 tetrahedra. $\square$

We can now estimate the norm of $\mathbf{\Psi}_E(\mathbf{v})$ by the following theorem,

Theorem 5.2. *For any $\mathbf{v} \in \widetilde{\mathcal{W}}_E$, there exists a constant $C > 0$ independent of h and $\mathbf{v}$ such that*

$$\|\mathbf{\Psi}_E(\mathbf{v})\|_{0,E}^2 \leqslant C \left(\int_E |\nabla \cdot \mathbf{v}| + \sum_{F \in \mathcal{F}_E^{\text{ext}}} \int_F |\mathbf{v} \cdot \mathbf{n}_F| \right)^2. \tag{11}$$

Proof. Let $\mathbf{v} \in \widetilde{\mathcal{W}}_E$. By using the norm equivalence (10), we have

$$\|\mathbf{\Psi}_E(\mathbf{v})\|_{0,E}^2 \leqslant \beta \sum_{i=1}^{n_E} [\![\mathbf{\Psi}_E(\mathbf{v})]\!]_{T_i}^2 \leqslant 4\beta \left(\sum_{F \in \widetilde{\mathcal{F}}_E} \int_F |\mathbf{\Psi}_E(\mathbf{v}) \cdot \mathbf{n}_F| \right)^2.$$

We deduce inequality (11) from the expression of normal components of $\mathbf{\Psi}_E(\mathbf{v})$ in (9). $\square$

5.2 Projection operators

It remains to bound the norm of $\mathbf{\Phi}_E(\mathbf{v})$, which is possible if $\mathbf{v}$ is the projection of a velocity in $\mathbf{H}^1(E)$. We define then the projection operators onto the approximation spaces.

Let π_h be the projection operator from $\mathcal{M} = L^2(\Omega)$ onto the approximation space $\mathcal{M}_h$ defined as

$$\pi_h(q)|_E = \pi_E(q), \quad \pi_E(q) = \frac{1}{|E|} \int_E q, \qquad \forall E \in \mathcal{T}_h, \quad \forall q \in \mathcal{M}.$$

Let $\mathbf{\Pi}_h$ be the projection operator from $\mathbf{H}^1(\Omega)$ onto the space $\mathcal{W}_h$ defined as

$$\mathbf{\Pi}_h(\mathbf{v})|_E = \mathbf{\Pi}_E(\mathbf{v}), \quad \mathbf{\Pi}_E(\mathbf{v}) = \sum_{F \in \mathcal{F}_E^{\text{ext}}} v_F \mathbf{w}_{E,F}, \quad v_F = \int_F \mathbf{v} \cdot \mathbf{n}_F, \qquad \forall E \in \mathcal{T}_h, \forall \mathbf{v} \in \mathbf{H}^1(\Omega) \tag{12}$$

with $\mathbf{w}_{E,F}, F \in \mathcal{F}_E^{\text{ext}}$, the basis functions of $\mathcal{W}_E$, solutions of problems $(\mathcal{P}_{E,F})$.

Similarly, the projection operators onto the approximation spaces $(\widetilde{\mathcal{M}}_h, \widetilde{\mathcal{W}}_h)$ for the **RTN** method are defined on the submesh $\widetilde{\mathcal{T}}_h$. Let $\widetilde{\pi}_h$ be the projection operator from $\mathcal{M} = L^2(\Omega)$ onto $\mathcal{M}_h$ defined as

$$\widetilde{\pi}_h(q)|_T = \widetilde{\pi}_T(q), \quad \widetilde{\pi}_T(q) = \frac{1}{|T|} \int_T q, \qquad \forall T \in \widetilde{\mathcal{T}}_h, \quad \forall q \in \mathcal{M}.$$

Let $\widetilde{\mathbf{\Pi}}_h$ be the projection operator from $\mathbf{H}^1(\Omega)$ onto $\widetilde{\mathcal{W}}_h$ defined as

$$\widetilde{\mathbf{\Pi}}_h(\mathbf{v})|_T = \widetilde{\mathbf{\Pi}}_T(\mathbf{v}), \quad \widetilde{\mathbf{\Pi}}_T(\mathbf{v}) = \sum_{F \in \mathcal{F}_T} v_F \mathbf{w}_F, \quad v_F = \int_F \mathbf{v} \cdot \mathbf{n}_F, \qquad \forall T \in \widetilde{\mathcal{T}}_h, \forall \mathbf{v} \in \mathbf{H}^1(\Omega)$$

where the basis functions $\mathbf{w}_F$ of the $\mathbf{RTN}_0$ method are associated with the edges $F \in \mathcal{F}_T$ of the triangle T.

We deduce some known results from the definition of the projection operators. It is known [4, 14] that the following results hold for constants $C > 0$ independent of h:

$$\text{For } q \in L^2(E) \qquad \|q - \pi_E(q)\|_{0,E} \leqslant C\|q\|_{0,E} \qquad \forall E \in \mathcal{T}_h. \tag{13}$$

$$\text{For } q \in L^2(T) \qquad \|q - \widetilde{\pi}_T(q)\|_{0,T} \leqslant C\|q\|_{0,T} \qquad \forall T \in \widetilde{\mathcal{T}}_h. \tag{14}$$

$$\text{For } q \in H^1(E) \qquad \|q - \pi_E(q)\|_{0,E} \leqslant Ch\|\nabla q\|_{0,E} \qquad \forall E \in \mathcal{T}_h. \tag{15}$$

$$\text{For } q \in H^1(T) \qquad \|q - \widetilde{\pi}_T(q)\|_{0,T} \leqslant Ch\|\nabla q\|_{0,T} \qquad \forall T \in \widetilde{\mathcal{T}}_h.$$

For vector functions, we have the following commutative properties:

$$\text{For } \mathbf{v} \in \mathbf{H}^1(E) \qquad \pi_E(\nabla\cdot \mathbf{v}) = \nabla\cdot \mathbf{\Pi}_E(\mathbf{v}), \qquad \forall E \in \mathcal{T}_h. \tag{16}$$

$$\text{For } \mathbf{v} \in \mathbf{H}^1(T) \qquad \widetilde{\pi}_T(\nabla\cdot \mathbf{v}) = \nabla\cdot \widetilde{\mathbf{\Pi}}_T(\mathbf{v}), \qquad \forall T \in \widetilde{\mathcal{T}}_h. \tag{17}$$

The interpolation errors for $\widetilde{\mathbf{\Pi}}_h$ are known and proven in [16, Theorem 6.3] or in [6, Proposition 2.5.1]:

$$\text{For } \mathbf{v} \in \mathbf{H}^1(E) \qquad \|\mathbf{v} - \widetilde{\mathbf{\Pi}}_h(\mathbf{v})\|_{0,E} \leqslant Ch|\mathbf{v}|_{1,E}, \qquad \forall E \in \mathcal{T}_h. \tag{18}$$

$$\text{For } \mathbf{v} \in \mathbf{H}^1(E) \qquad \|\nabla\cdot \mathbf{v} - \nabla\cdot \widetilde{\mathbf{\Pi}}_h(\mathbf{v})\|_{0,E} \leqslant Ch|\nabla\cdot \mathbf{v}|_{1,E}, \qquad \forall E \in \mathcal{T}_h.$$

The interpolation errors for $\mathbf{\Pi}_h$ will be proven in Section 6.2.

5.3 A bound of the rotating velocity

If $\mathbf{v} \in \mathbf{H}^1(E)$, the velocity $\mathbf{\Phi}_E(\widetilde{\mathbf{\Pi}}_h(\mathbf{v}))$ can be interpreted as an interpolation error because these normal components at the boundary of E are zero. Consequently, its norm can be bounded like the estimate (18). We recall first the Bramble-Hilbert lemma in [7, Theorem 2], which is used to bound the interpolation errors. Later we give an estimate of $\|\mathbf{\Phi}_E(\widetilde{\mathbf{\Pi}}_h(\mathbf{v}))\|_{0,E}$.

Lemma 5.3 (Bramble-Hilbert)**.** *Let $E \subset \Omega$ be a Lipschitz domain. If the linear operator $\mathcal{F}\colon \mathbf{H}^1(E) \to \mathbb{R}$ meets the following conditions for any $\mathbf{v} \in \mathbf{H}^1(E)$:*

1. $\|\mathcal{F}(\mathbf{v})\|_{0,E} \leqslant C\|\mathbf{v}\|_{1,E}$,
2. $\mathcal{F}(\mathbf{v}) = 0$ *when* $\mathbf{v}$ *is constant,*

then there exists a constant $C > 0$ independent of h and $\mathbf{v}$ such that

$$\|\mathcal{F}(\mathbf{v})\|_{0,E} \leqslant Ch\|\mathbf{v}\|_{1,E}.$$

Theorem 5.4. *Let $\mathbf{v} \in \mathbf{H}^1(E)$. There exists a constant $C > 0$ independent of h and $\mathbf{v}$ such that*

$$\|\boldsymbol{\Phi}_E(\widetilde{\boldsymbol{\Pi}}_h(\mathbf{v}))\|_{0,E} \leqslant Ch\|\mathbf{v}\|_{1,E}. \tag{19}$$

Proof. We prove the estimate of Theorem 5.4, by showing that on $E \in \mathcal{T}_h$, the operator $\mathcal{F}_E := \boldsymbol{\Phi}_E \circ \widetilde{\boldsymbol{\Pi}}_h$ satisfies the conditions of Lemma 5.3. For $\mathbf{v} \in \mathbf{H}^1(E)$ we have

$$\|\mathcal{F}_E(\mathbf{v})\|_{0,E} \leqslant \|\mathbf{v}\|_{0,E} + \|\boldsymbol{\Psi}_E(\widetilde{\boldsymbol{\Pi}}_h(\mathbf{v}))\|_{0,E} \leqslant C\|\mathbf{v}\|_{1,E},$$

by using the decomposition of $\mathbf{v}$ in (8), the estimate (11) and the norm equivalence in (10).

Remains to prove the condition $\mathcal{F}(\mathbf{v}) = 0$ when $\mathbf{v}$ is constant. As shown in Figure 2, we place ourselves into the coordinate system centered on m_E, where the x axis is oriented to follow $\mathbf{v}$. For $i = 1, \ldots, n_E$, the different vertices of E are denoted $p_i = \left(\begin{smallmatrix} x_i \\ y_i \end{smallmatrix}\right)$. We deduce that $\int_{F_i^{\text{int}}} \mathbf{v} \cdot \mathbf{n}_{F_i^{\text{int}}} = -\|\mathbf{v}\| y_i$ because $\mathbf{v}$ follows the x axis and $\nabla \cdot \mathbf{v} = 0$. Consequently,

$$\phi_E(\mathbf{v}) = -\frac{\|\mathbf{v}\|}{n_E} \sum_{i=1}^{n_E} y_i = 0,$$

because m_E is the barycenter of the vertices of E. $\square$

6 A priori error estimation

From the previous results, we can now prove the convergence of the composite method. First, we estimate the norm of the projection $\boldsymbol{\Pi}_E(\mathbf{v})$, needed to have the estimate (5). Then we prove the convergence of the composite method.

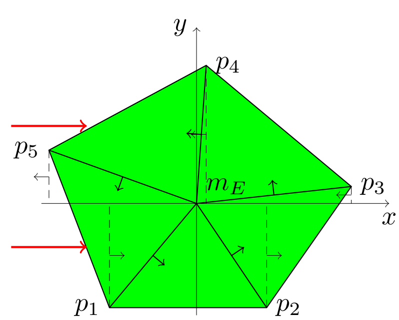

Figure 2: The mesh $\mathcal{T}_E$ with a constant velocity

6.1 A bound of the projection operator for the velocities

From a trace theorem [10, Theorem 1.5.1.10], there exists a constant $C > 0$ independent of h such that,

$$\left|\int_F \mathbf{v}\cdot\mathbf{n}_F\right| \leqslant C\|\mathbf{v}\|_{1,E} \qquad \forall F \in \mathcal{F}_E, \quad \forall \mathbf{v} \in \mathbf{H}^1(E), \quad \forall E \in \mathcal{T}_h. \tag{20}$$

With this result, the norm of the velocities projected by $\mathbf{\Pi}_h$ can be estimated, which is necessary to prove the convergence of the method.

Theorem 6.1. *For velocities $\mathbf{v} \in \mathbf{H}^1(\Omega)$, there exists a constant $C > 0$ independent of h such that*

$$\|\mathbf{\Pi}_h(\mathbf{v})\|_{\mathbf{H}(\mathrm{div},\Omega)} \leqslant C\|\mathbf{v}\|_{1,\Omega}. \tag{21}$$

Proof. Let $\mathbf{v} \in \mathbf{H}^1(\Omega)$. The norm is studied on each element $E \in \mathcal{T}_h$. By definition of the projection operator (12), we have

$$\|\mathbf{\Pi}_E(\mathbf{v})\|_{0,E} \leqslant \sum_{F\in\mathcal{F}_E}\left|\int_F \mathbf{v}\cdot\mathbf{n}_F\right|\|\mathbf{w}_{E,F}\|_{0,E}.$$

For an edge $F \in \mathcal{F}_E$, the basis function $\mathbf{w}_{E,F}$ solves the problem $(\mathcal{P}_{E,F})$. From it, we deduce that the norm of $\mathbf{w}_{E,F}$ can be bounded by (11), by using the decomposition (8) of velocities, and the property $\phi_E(\mathbf{w}_{E,F}) = 0$. There exists then a constant $C > 0$ such that

$$\|\mathbf{w}_{E,F}\|_{0,E}^2 \leqslant C\left(\int_E|\nabla\cdot\mathbf{w}_{E,F}| + \sum_{F'\in\mathcal{F}_E}\int_{F'}|\mathbf{w}_{E,F}\cdot\mathbf{n}_{F'}|\right)^2.$$

Moreover, we have $\nabla\cdot\mathbf{w}_{E,F} = |E|^{-1}$ and $\mathbf{w}_{E,F}\cdot\mathbf{n}_{F'} = |F|^{-1}\delta_{FF'}$, for an edge $F' \in \mathcal{F}_E$. Consequently,

$$\|\mathbf{\Pi}_E(\mathbf{v})\|_{0,E} \leqslant C\sum_{F\in\mathcal{F}_E}\left|\int_F \mathbf{v}\cdot\mathbf{n}_F\right|,$$

and from (20), we obtain $\|\mathbf{\Pi}_E(\mathbf{v})\|_{0,E} \leqslant C\|\mathbf{v}\|_{1,E}$.

Concerning the divergence of $\mathbf{\Pi}_E(\mathbf{v})$, using the commutativity property (16) we obtain

$$\begin{aligned}\|\nabla\cdot\mathbf{\Pi}_E(\mathbf{v})\|_{0,E} &\leqslant \|\nabla\cdot\mathbf{\Pi}_E(\mathbf{v}) - \nabla\cdot\mathbf{v}\|_{0,E} + \|\nabla\cdot\mathbf{v}\|_{0,E}\\ &\leqslant \|\pi_E(\nabla\cdot\mathbf{v}) - \nabla\cdot\mathbf{v}\|_{0,E} + \|\nabla\cdot\mathbf{v}\|_{0,E}.\end{aligned}$$

We conclude by using inequality (13). □

We can prove that the approximation spaces $\mathcal{M}_h$ and $\mathcal{W}_h$ satisfy the conditions (4a) and (4b), and so prove the estimate (5) with the previous results. The first condition (4a) on the bilinear form a holds using (3a) and the fact that $\mathcal{V}_h$ is a subset of $\mathcal{V}$.

To prove (4b), another condition is shown from (3b) in [16, Theorem 13.2]. The bilinear form b satisfies the inf-sup condition with respect to the spaces $\mathcal{M}_h$ and $\mathbf{H}^1(\Omega)$,

$$\exists \beta > 0, \quad \inf_{q_h \in \mathcal{M}_h} \sup_{\mathbf{v} \in \mathbf{H}^1(\Omega)} b(\mathbf{v}, q_h) \geqslant \beta \|\mathbf{v}\|_{1,\Omega} \|q_h\|_{0,\Omega}.$$

Consequently, for $q_h \in \mathcal{M}_h$, there exists a velocity $\mathbf{v} \in \mathbf{H}^1(\Omega)$ such that

$$b(\mathbf{v}, q_h) \geqslant \beta \|\mathbf{v}\|_{\mathbf{H}^1(\Omega)} \|q_h\|_{0,\Omega}.$$

Since $b(\mathbf{\Pi}_h(\mathbf{v}), q_h) = b(\mathbf{v}, q_h)$ and since the velocity is bounded by (21), we have

$$b(\mathbf{\Pi}_h(\mathbf{v}), q_h) \geqslant \beta C \|\mathbf{\Pi}_h(\mathbf{v})\|_{\mathbf{H}(\mathrm{div},\Omega)} \|q_h\|_{0,\Omega}.$$

This inequality holds for a velocity $\mathbf{v}_h = \mathbf{\Pi}_h(\mathbf{v}) \in \mathcal{W}_h$, so for the supremum, and for all functions $q_h \in \mathcal{M}_h$, which proves the estimate (5).

6.2 Error estimates

It remains to estimate the interpolation errors for the composite method.

Theorem 6.2 (Interpolation errors)**.** *Let $\mathbf{u} \in \mathcal{W}$, $p \in \mathcal{M}$ be the solution of problem $(\mathcal{P})$. If $\mathbf{u} \in \mathbf{H}^1(\Omega)$ and $\nabla\cdot \mathbf{u} \in L^2(\Omega)$, then there exist constants $C > 0$ independent of h such that*

$$\|p - \pi_h(p)\|_{0,\Omega} \leqslant Ch|p|_{1,\Omega}, \tag{22}$$

$$\|\mathbf{u} - \mathbf{\Pi}_h(\mathbf{u})\|_{0,\Omega} \leqslant Ch\|\mathbf{u}\|_{1,\Omega}, \tag{23}$$

$$\|\nabla\cdot(\mathbf{u} - \mathbf{\Pi}_h(\mathbf{u}))\|_{0,\Omega} \leqslant Ch\|\nabla\cdot \mathbf{u}\|_{1,\Omega}. \tag{24}$$

Proof. The estimate of the interpolation error for the scalar functions (22) is just inequality (15) extended to Ω.

For the interpolation error of the vector functions (23) we have

$$\|\mathbf{u} - \mathbf{\Pi}_h(\mathbf{u})\|_{0,\Omega} \leqslant \|\mathbf{u} - \widetilde{\mathbf{\Pi}}_h(\mathbf{u})\|_{0,\Omega} + \|\widetilde{\mathbf{\Pi}}_h(\mathbf{u}) - \mathbf{\Pi}_h(\mathbf{u})\|_{0,\Omega}.$$

Estimate (18) gives a bound on $\|\mathbf{u} - \widetilde{\mathbf{\Pi}}_h(\mathbf{u})\|_{0,\Omega}$. To bound $\|\widetilde{\mathbf{\Pi}}_h(\mathbf{u}) - \mathbf{\Pi}_h(\mathbf{u})\|_{0,\Omega}$, we use the decomposition of velocities in (8) on a polygon $E \in \mathcal{T}_h$. We also remark that $\phi_E(\mathbf{\Pi}_h(\mathbf{u})) = 0$, since it is a linear combination of the basis functions, which are solutions of problems $(\mathcal{P}_{E,F})$. Therefore we obtain

$$\|\widetilde{\mathbf{\Pi}}_h(\mathbf{u}) - \mathbf{\Pi}_h(\mathbf{u})\|_{0,E} \leqslant \|\mathbf{\Phi}_E(\widetilde{\mathbf{\Pi}}_h(\mathbf{u}))\|_{0,E} + \|\mathbf{\Psi}_E(\widetilde{\mathbf{\Pi}}_h(\mathbf{u}) - \mathbf{\Pi}_h(\mathbf{u}))\|_{0,E}.$$

Since $\mathbf{u} \in \mathbf{H}^1(E)$, the rotating velocity $\mathbf{\Phi}_E(\widetilde{\mathbf{\Pi}}_h(\mathbf{u}))$ is bounded by (19). The remainder is bounded by (11), which gives

$$\|\mathbf{\Psi}_E(\widetilde{\mathbf{\Pi}}_h(\mathbf{u}) - \mathbf{\Pi}_h(\mathbf{u}))\|_{0,E} \leqslant C \left(\int_E \left| \nabla\cdot \left(\widetilde{\mathbf{\Pi}}_h(\mathbf{u}) - \mathbf{\Pi}_h(\mathbf{u}) \right) \right| + \sum_{F \in \mathcal{F}_E} \int_F \left| \left(\widetilde{\mathbf{\Pi}}_h(\mathbf{u}) - \mathbf{\Pi}_h(\mathbf{u}) \right) \cdot \mathbf{n}_F \right| \right),$$

with $C > 0$ independent of h and $\mathbf{u}$. On an edge $F \in \mathcal{F}_E$ at the boundary, $\mathbf{\Pi}_h(\mathbf{u})|_F = \widetilde{\mathbf{\Pi}}_h(\mathbf{u})|_F$, so

$$\int_F \left| \left(\widetilde{\mathbf{\Pi}}_h(\mathbf{u}) - \mathbf{\Pi}_h(\mathbf{u}) \right) \cdot \mathbf{n}_F \right| = 0.$$

Concerning the estimate of divergence in the sum, the Cauchy-Schwarz inequality used together with the commutative properties (16) and (17), and inequalities (13) and (14) gives:

$$\begin{aligned}
\int_E \left| \nabla\cdot \left(\widetilde{\mathbf{\Pi}}_h(\mathbf{u}) - \mathbf{\Pi}_h(\mathbf{u}) \right) \right| &\leqslant |E|^{1/2} \left\| \nabla\cdot \left(\widetilde{\mathbf{\Pi}}_h(\mathbf{u}) - \mathbf{\Pi}_h(\mathbf{u}) \right) \right\|_{0,E} \\
&\leqslant h \left(\|\widetilde{\pi}_h\left(\nabla\cdot \mathbf{u}\right) - \nabla\cdot \mathbf{u}\|_{0,E} + \|\nabla\cdot \mathbf{u} - \pi_h\left(\nabla\cdot \mathbf{u}\right)\|_{0,E} \right) \\
&\leqslant Ch \|\nabla\cdot \mathbf{u}\|_{0,E}
\end{aligned}$$

where C is a constant independent of h.

Finally to prove inequality (24), the commutativity property (16) together with inequality (15) imply

$$\|\nabla\cdot (\mathbf{u} - \mathbf{\Pi}_h(\mathbf{u}))\|_{0,E} = \|\nabla\cdot \mathbf{u} - \pi_h(\nabla\cdot \mathbf{u})\|_{0,E} \leqslant Ch\|\nabla\cdot \mathbf{u}\|_{1,E}$$

where C is a constant independent of h. This concludes the proof of Theorem 6.2. □

Theorem 6.3. *Let $\mathbf{u} \in \mathcal{W}$, $p \in \mathcal{M}$ be the solution of problem $(\mathcal{P})$, and $\mathbf{u}_h \in \mathcal{W}_h$, $p_h \in \mathcal{M}_h$ be the solution of $(\mathcal{P}_h)$. If $\mathbf{u} \in \mathbf{H}^1(\Omega)$, then there exists a constant $C > 0$ independent of h such that*

$$\|p - p_h\|_{0,\Omega} + \|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{H}(\mathrm{div},\Omega)} \leqslant Ch \left(|p|_{1,\Omega} + \|\mathbf{u}\|_{1,\Omega} + \|\nabla\cdot \mathbf{u}\|_{1,\Omega} \right).$$

Proof. The error of the convergence is proved by using the error estimate (5) and the interpolation errors (22), (23) and (24). □

7 Numerical experiments

The convergence of the composite method is shown on the domain $\Omega = [0\,;1]^2$ with meshes of quadrangles. n denotes the number of discretization intervals in each direction. The exact scalar solution to compute is

$$p(x,y) = \frac{x^3}{2} + xy^2.$$

With an anisotropic tensor $\mathbf{K} = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 20 \end{smallmatrix}\right)$, the expression of the exact velocity $\mathbf{u}$ is:

$$\mathbf{u}(x,y) = -\begin{pmatrix} 3x^2 + 2y^2 + 2xy \\ \frac{3}{2}x^2 + y^2 + 40xy \end{pmatrix}.$$

The first numerical experiment is performed on rectangular meshes, shown in Figure 3. The convergence errors and orders of convergence of the composite method are shown in Table 1, and compared with the errors of the $\mathbf{RTN}_0$ method on the corresponding triangular submesh in Table 2 and one can check that the two methods converge with the same rate.

The second numerical experiment uses non rectangular meshes shown in Figure 4. These meshes are not built by refining a coarse mesh, so all meshes maintain the same aspect ratio for the quadrangles. Even on this kind of meshes, the method converges with an optimal rate as shown in Table 3, even though the errors are a little larger than that of the $\mathbf{RTN}_0$ method on the triangular submeshes shown in Table 4.

Note that in both experiments the triangular $\mathbf{RTN}_0$ method uses for velocity 3 times as many degrees of freedom as the composite method and 4 times as many for pressure.

8 Conclusion

We constructed a two-dimensional composite mixed finite element for polygonal meshes by adding an interior point to the polygonal cell which serves as a vertex as well as the polygon vertices for a triangular submesh of the polygon. We analyzed the method and showed optimal convergence. This convergence was confirmed by numerical experiments. This analysis is a first step towards the analysis of a 3-D composite mixed finite element with one interior point inside the element [5].

(a) for $n = 8$ **(b)** for $n = 16$

Figure 3: Rectangular meshes.

n	h	$\|p - p_h\|_{0,\Omega}$ error	rate	$\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega}$ error	rate
2	$7.07 \cdot 10^{-1}$	$1.66 \cdot 10^{-1}$		$3.79 \cdot 10^{0}$	
4	$3.54 \cdot 10^{-1}$	$8.55 \cdot 10^{-2}$	0.96	$1.91 \cdot 10^{0}$	0.99
8	$1.77 \cdot 10^{-1}$	$4.30 \cdot 10^{-2}$	0.99	$9.57 \cdot 10^{-1}$	1.00
16	$8.84 \cdot 10^{-2}$	$2.15 \cdot 10^{-2}$	1.00	$4.79 \cdot 10^{-1}$	1.00
32	$4.42 \cdot 10^{-2}$	$1.08 \cdot 10^{-2}$	1.00	$2.39 \cdot 10^{-1}$	1.00
64	$2.21 \cdot 10^{-2}$	$5.39 \cdot 10^{-3}$	1.00	$1.20 \cdot 10^{-1}$	1.00
128	$1.10 \cdot 10^{-2}$	$2.69 \cdot 10^{-3}$	1.00	$5.98 \cdot 10^{-2}$	1.00
256	$5.52 \cdot 10^{-3}$	$1.35 \cdot 10^{-3}$	1.00	$2.99 \cdot 10^{-2}$	1.00

Table 1: Error of the composite method on rectangular meshes.

n	h	$\|p - p_h\|_{0,\Omega}$ error	rate	$\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega}$ error	rate
2	$5.00 \cdot 10^{-1}$	$1.03 \cdot 10^{-1}$		$3.85 \cdot 10^{0}$	
4	$2.50 \cdot 10^{-1}$	$5.01 \cdot 10^{-2}$	1.03	$1.95 \cdot 10^{0}$	0.98
8	$1.25 \cdot 10^{-1}$	$2.49 \cdot 10^{-2}$	1.01	$9.77 \cdot 10^{-1}$	1.00
16	$6.25 \cdot 10^{-2}$	$1.25 \cdot 10^{-2}$	1.00	$4.89 \cdot 10^{-1}$	1.00
32	$3.13 \cdot 10^{-2}$	$6.22 \cdot 10^{-3}$	1.00	$2.44 \cdot 10^{-1}$	1.00
64	$1.56 \cdot 10^{-2}$	$3.11 \cdot 10^{-3}$	1.00	$1.22 \cdot 10^{-1}$	1.00
128	$7.81 \cdot 10^{-3}$	$1.56 \cdot 10^{-3}$	1.00	$6.11 \cdot 10^{-2}$	1.00
256	$3.91 \cdot 10^{-3}$	$7.78 \cdot 10^{-4}$	1.00	$3.06 \cdot 10^{-2}$	1.00

Table 2: Error of the $\mathbf{RTN}_0$ method on the triangular submeshes of the rectangular meshes.

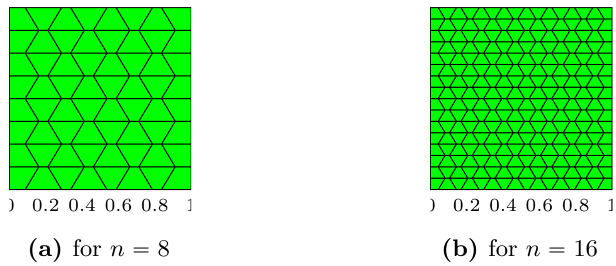

Figure 4: Non rectangular meshes with fixed aspect ratio.

		$\|p - p_h\|_{0,\Omega}$		$\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega}$	
n	h	error	rate	error	rate
2	$8.20 \cdot 10^{-1}$	$1.63 \cdot 10^{-1}$		$5.11 \cdot 10^{0}$	
4	$4.10 \cdot 10^{-1}$	$9.07 \cdot 10^{-2}$	0.85	$3.20 \cdot 10^{0}$	0.67
8	$2.05 \cdot 10^{-1}$	$4.65 \cdot 10^{-2}$	0.96	$1.61 \cdot 10^{0}$	0.99
16	$1.03 \cdot 10^{-1}$	$2.35 \cdot 10^{-2}$	0.98	$7.79 \cdot 10^{-1}$	1.05
32	$5.13 \cdot 10^{-2}$	$1.18 \cdot 10^{-2}$	0.99	$3.81 \cdot 10^{-1}$	1.03
64	$2.56 \cdot 10^{-2}$	$5.91 \cdot 10^{-3}$	1.00	$1.88 \cdot 10^{-1}$	1.02
128	$1.28 \cdot 10^{-2}$	$2.96 \cdot 10^{-3}$	1.00	$9.34 \cdot 10^{-2}$	1.01
256	$6.41 \cdot 10^{-3}$	$1.48 \cdot 10^{-3}$	1.00	$4.65 \cdot 10^{-2}$	1.01

Table 3: Error of the composite method on non rectangular meshes with fixed aspect ratio.

		$\|p - p_h\|_{0,\Omega}$		$\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega}$	
n	h	error	rate	error	rate
2	$6.50 \cdot 10^{-1}$	$1.12 \cdot 10^{-1}$		$4.88 \cdot 10^{0}$	
4	$4.00 \cdot 10^{-1}$	$6.63 \cdot 10^{-2}$	0.76	$3.10 \cdot 10^{0}$	0.66
8	$2.00 \cdot 10^{-1}$	$3.30 \cdot 10^{-2}$	1.01	$1.58 \cdot 10^{0}$	0.97
16	$1.00 \cdot 10^{-1}$	$1.63 \cdot 10^{-2}$	1.02	$7.74 \cdot 10^{-1}$	1.03
32	$5.00 \cdot 10^{-2}$	$8.13 \cdot 10^{-3}$	1.00	$3.80 \cdot 10^{-1}$	1.02
64	$2.50 \cdot 10^{-2}$	$4.08 \cdot 10^{-3}$	0.99	$1.88 \cdot 10^{-1}$	1.01
128	$1.25 \cdot 10^{-2}$	$2.05 \cdot 10^{-3}$	1.00	$9.36 \cdot 10^{-2}$	1.01
256	$6.25 \cdot 10^{-3}$	$1.03 \cdot 10^{-3}$	1.00	$4.67 \cdot 10^{-2}$	1.00

Table 4: Error of the $\mathbf{RTN}_0$ method on non rectangular meshes with fixed aspect ratio.

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